

# The Best Constant, the Existence of Extremal Functions and Related Results for an Improved Hardy-Sobolev Inequality

N. B. Zographopoulos

Department of Science, Division of Mathematics,  
Technical University of Crete,  
73100 Chania, Greece  
E-mail address: nzogr@science.tuc.gr

## Abstract

We present the best constant and the existence of extremal functions for an Improved Hardy-Sobolev inequality. We prove that, under a proper transformation, this inequality is equivalent to the Sobolev inequality in  $\mathbb{R}^N$ . We also discuss the connection of the related functional spaces and as a result we obtain some Caffarelli - Kohn - Nirenberg inequalities. Our starting point is the existence of a minimizer for the Bliss' inequality and the indirect dependence of the Hardy inequality at the origin.

*Keywords: Best Constants, Extremal Functions, Hardy-Sobolev inequality, Sobolev inequality, Bliss' inequality, Caffarelli - Kohn - Nirenberg Inequalities*

## 1 Introduction

Assume the following inequality:

$$\int_0^R r |v'|^2 dr \geq c \left( \int_0^R r^{-1} \left( -\log \left( \frac{r}{R} \right) \right)^{-\frac{2(N-1)}{N-2}} |v|^{\frac{2N}{N-2}} dr \right)^{\frac{N-2}{N}}, \quad (1.1)$$

which holds for any function  $v \in C_0^\infty(0, R)$ . This inequality may be obtained from a more general inequality [29, Theorem 4] (see also [25, Lemma 2.2]). However, as prof. V. Maz'ya pointed to us this inequality is also obtained from Bliss' inequality [10] (For the derivation of this inequality and some related discussion we refer to Section 3). In this work, we prove that under a proper transformation inequality (1.1) is equivalent to the Sobolev inequality in  $\mathbb{R}^N$  and consequently we obtain the best constants and the minimizers for (1.1). The best constant in the Sobolev inequality in  $\mathbb{R}^N$ :

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx \geq S \left( \int_{\mathbb{R}^N} |u|^{\frac{2N}{N-2}} dx \right)^{\frac{N-2}{N}}, \quad (1.2)$$

as it is well known, see [6, 28, 33], is

$$S(N) = \frac{N(N-2)}{4} |\mathbb{S}_N|^{2/N} = 2^{2/N} \pi^{1+1/N} \Gamma \left( \frac{N+1}{2} \right)^{-2/N},$$

where  $\mathbb{S}_N$  is the area of the  $N$ -dimensional unit sphere and the extremal functions are

$$\psi_{\mu,\nu}(|x|) = (\mu^2 + \nu^2|x|)^{-(N-2)/2}, \quad \mu \neq 0, \nu \neq 0.$$

For a quantitative version of the sharp Sobolev inequality we refer to [20].

**LEMMA 1.1** *Inequality (1.1) under the transformation*

$$u(r) = w(t), \quad t = \left(-\log\left(\frac{r}{R}\right)\right)^{-\frac{1}{N-2}} \quad (1.3)$$

is equivalent to (1.2). The best constant is

$$C_M = (N-2)^{-\frac{2(N-1)}{N}} (N\omega_N)^{-\frac{2}{N}} S(N) = \frac{1}{4} \left(\frac{N}{N-2}\right)^{\frac{N-2}{N}} \left(\frac{|\mathbb{S}_N|}{\omega_N}\right)^{\frac{2}{N}}, \quad (1.4)$$

where  $\omega_N$  denotes the Lebesgue measure of the unit ball in  $\mathbb{R}^N$  and the minimizers are

$$\phi_{\mu,\nu}(r) = \psi_{\mu,\nu}(t) = \left(\mu^2 + \nu^2 \left(-\log\left(\frac{r}{R}\right)\right)^{-\frac{2}{N-2}}\right)^{-\frac{N-2}{2}}, \quad \mu \neq 0, \nu \neq 0. \quad (1.5)$$

It is clear that  $\phi$  may be continuously defined as  $\phi_{\mu,\nu}(0) = \mu^{-(N-2)}$  and  $\phi_{\mu,\nu}(R) = 0$ .

As an application of inequality (1.1) the authors in [25], proved the following Improved Hardy-Sobolev (IHS) inequality:

$$\begin{aligned} \int_{B_R} |\nabla u(|x|)|^2 dx &\geq \left(\frac{N-2}{2}\right)^2 \int_{B_R} \frac{u^2(|x|)}{|x|^2} dx \\ &\quad + C_{HS} \left( \int_{B_R} |u(|x|)|^{\frac{2N}{N-2}} \left(-\log\left(\frac{|x|}{R}\right)\right)^{-\frac{2(N-1)}{N-2}} dx \right)^{\frac{N-2}{N}}, \end{aligned} \quad (1.6)$$

in the radial case, i.e. where  $B_R$  is the open ball in  $\mathbb{R}^N$ ,  $N \geq 3$ , of radius  $R$  centered at the origin and  $u \in C_0^\infty(B_R \setminus \{0\})$  is a radially symmetric function. The same result was proved in [30], with the use of a Caffarelli-Kohn-Nirenberg inequality. Actually, in [25] the following general (not in necessarily radial case) IHS inequality was proved: Let  $\Omega$  be a bounded domain in  $\mathbb{R}^N$ ,  $N \geq 3$ , containing the origin,  $D_0 = \sup_{x \in \Omega} |x|$  and  $D > D_0$ , then the following inequality

$$\begin{aligned} \int_{\Omega} |\nabla u|^2 dx &\geq \left(\frac{N-2}{2}\right)^2 \int_{\Omega} \frac{u^2}{|x|^2} dx \\ &\quad + C_{HS}(\Omega) \left( \int_{\Omega} |u|^{\frac{2N}{N-2}} \left(-\log\left(\frac{|x|}{D}\right)\right)^{-\frac{2(N-1)}{N-2}} dx \right)^{\frac{N-2}{N}} \end{aligned} \quad (1.7)$$

holds for any  $u \in C_0^\infty(\Omega \setminus \{0\})$ . From the discussion in [25, 30], it is clear that the nature of (1.7) depends on the distance of  $D$  from  $D_0$ , for instance in the case where  $D = D_0$  the author in [30] proved that the inequality cannot hold if we consider nonradial functions.

Both papers follow the approach that is based on the following change of variables (This approach was introduced in [13] and followed in various ways by many authors); For any  $u \in H_0^1(\Omega)$  we set

$$u = |x|^{\frac{N-2}{2}} v \quad (1.8)$$

and in this case we have

$$\int_{\Omega} |\nabla u|^2 dx - \left( \frac{N-2}{2} \right)^2 \int_{\Omega} \frac{u^2}{|x|^2} dx = \int_{\Omega} |x|^{-(N-2)} |\nabla v|^2 dx. \quad (1.9)$$

Then, inequality (1.7) is equivalent to

$$\int_{\Omega} |x|^{-(N-2)} |\nabla v|^2 dx \geq C_{HS}(\Omega) \left( \int_{\Omega} |x|^{-N} |v|^{\frac{2N}{N-2}} \left( -\log \left( \frac{|x|}{D} \right) \right)^{-\frac{2(N-1)}{N-2}} dx \right)^{\frac{N-2}{N}} \quad (1.10)$$

and which in turn is equivalent, in the radial case, to (1.1). Therefore, is natural to consider the space  $W_0^{1,2}(|x|^{-(N-2)}, \Omega)$ , see [25], which is defined as the complement of the  $C_0^\infty(\Omega)$  functions under the norm

$$\|v\|_{W_0^{1,2}(|x|^{-(N-2)}, \Omega)}^2 = \int_{\Omega} |x|^{-(N-2)} |\nabla v|^2 dx.$$

The space  $W_0^{1,2}(|x|^{-(N-2)}, \Omega)$  has the property (for a generalization see Lemma 1.2) that if  $u \in H_0^1(\Omega)$  then  $|x|^{(N-2)/2} u \in W_0^{1,2}(|x|^{-(N-2)}, \Omega)$ . (Some other properties of this space may be found in Section 2). The advantage of this space is the following; assume an inequality, e.g. an improved Hardy inequality (see [25, Section 3]), which admits no  $H_0^1$ - minimizer then, under the change of variables (1.8), the corresponding inequality admits  $W_0^{1,2}(|x|^{-(N-2)}, \Omega)$  minimizer. This happens because if  $v \in W_0^{1,2}(|x|^{-(N-2)}, \Omega)$  then it is not necessary that  $|x|^{-(N-2)/2} v$  belongs in  $H_0^1(\Omega)$ .

From (1.9) it is also natural to define the space  $H$  as the completion of the set

$$\left\{ |x|^{-\frac{N-2}{2}} \phi(x); \phi \in C_0^\infty(\Omega) \right\}$$

under the norm

$$\|u\|_{H(\Omega)}^2 = \int_{\Omega} |\nabla u|^2 dx - \left( \frac{N-2}{2} \right)^2 \int_{\Omega} \frac{u^2}{|x|^2} dx - L^2(u) \quad (1.11)$$

where  $u_r$  is the radial part of  $u$ , i.e. we extend  $u$  as zero outside  $\Omega$ , and for some  $R > \sup_{x \in \Omega} |x|$ , we take the projection of  $u$  on the space of radially symmetric functions, i.e.,

$$u_r(|x|) = \frac{1}{|\partial B_R|} \int_{\partial B_R} u ds$$

and by  $L(u)$  we denote the quantity

$$L(u) := \left( \frac{N(N-2)}{2} \omega_N \right)^{1/2} \lim_{|x| \rightarrow 0} |x|^{\frac{N-2}{2}} u_r(|x|). \quad (1.12)$$

For the definition of this space and some related properties we refer to [36, 37]. We note that  $H_0^1(\Omega)$  is a subspace of  $H(\Omega)$ . The fact that the space  $H$  is not convenient to be defined as the completion of the  $C_0^\infty(\Omega)$  functions under the norm

$$\|\phi\|_{H(\Omega)}^2 = \int_{\Omega} |\nabla \phi|^2 dx - \left( \frac{N-2}{2} \right)^2 \int_{\Omega} \frac{\phi^2}{|x|^2} dx. \quad (1.13)$$

is explained in [36] and this due to the presence of a “boundary” term; if we define  $H$  with norm given by (1.13) then functions that behave at the origin like  $|x|^{-(N-2)/2}$  fail to be in  $H$ .

The connection between the spaces  $H(\Omega)$  and  $W_0^{1,2}(|x|^{-(N-2)}, \Omega)$  is given in the following lemma;

**LEMMA 1.2** *Assume that  $\Omega$  is a bounded domain of  $\mathbb{R}^N$ ,  $N \geq 3$ , containing the origin. Then,  $u \in H(\Omega)$  if and only if  $|x|^{(N-2)/2} u \in W_0^{1,2}(|x|^{-(N-2)}, \Omega)$ . In this case the connection of the norms is given by*

$$\|u\|_{H(\Omega)}^2 = \|v\|_{W_0^{1,2}(|x|^{-(N-2)}, \Omega)}^2,$$

or

$$\int_{\Omega} |\nabla u|^2 dx - \left(\frac{N-2}{2}\right)^2 \int_{\Omega} \frac{u^2}{|x|^2} dx - L^2(u) = \int_{\Omega} |x|^{-(N-2)} |\nabla v|^2 dx. \quad (1.14)$$

In addition we can relate these spaces, in the radial case, with the space  $D^{1,2}(\mathbb{R}^N)$ , which is defined as the closure of  $C_0^\infty(\mathbb{R}^N)$  functions under the norm

$$\|\phi\|_{D^{1,2}(\mathbb{R}^N)}^2 = \int_{\mathbb{R}^N} |\nabla \phi|^2 dx.$$

For more details we refer to the classical book [1]. If we denote by  $H_r(\Omega)$ ,  $W_{0,r}^{1,2}(|x|^{-(N-2)}, \Omega)$  and  $D_r^{1,2}(\mathbb{R}^N)$  the subspaces of  $H(\Omega)$ ,  $W_0^{1,2}(|x|^{-(N-2)}, \Omega)$  and  $D^{1,2}(\mathbb{R}^N)$ , respectively, which consist of radial functions, we have that

**LEMMA 1.3** *Let  $v \in W_{0,r}^{1,2}(|x|^{-(N-2)}, B_R)$  and set*

$$v(|x|) = w(t), \quad t = \left(-\log\left(\frac{|x|}{R}\right)\right)^{-\frac{1}{N-2}} \quad (1.15)$$

*as in (1.3). Then,  $v \in W_{0,r}^{1,2}(|x|^{-(N-2)}, B_R)$  if and only if  $w \in D_r^{1,2}(\mathbb{R}^N)$  and*

$$\|v\|_{W_{0,r}^{1,2}(|x|^{-(N-2)}, B_R)} = (N-2)^{-1} \|w\|_{D_r^{1,2}(\mathbb{R}^N)}. \quad (1.16)$$

Observe that (1.16) is independent of the radius  $R$  and in the case where  $N = 3$  the norm in  $W_{0,r}^{1,2}(|x|^{-(N-2)}, B_R)$  coincides with the norm in  $D_r^{1,2}(\mathbb{R}^N)$ . Moreover, (1.14) and (1.16) imply that

**COROLLARY 1.1** *Let  $u \in H_r(B_R)$  and set*

$$w(t) = |x|^{\frac{N-2}{2}} u(|x|), \quad t = \left(-\log\left(\frac{|x|}{R}\right)\right)^{-\frac{1}{N-2}}. \quad (1.17)$$

*Then, if  $u \in H_r(B_R)$  then  $w \in D_r^{1,2}(\mathbb{R}^N)$  and*

$$\|u\|_{H_r(B_R)}^2 = (N-2)^{-1} \|w\|_{D_r^{1,2}(\mathbb{R}^N)}^2. \quad (1.18)$$

For a related to the IHS inequality (1.6), as a consequence of Lemma 1.1, we have

**THEOREM 1.1** *The infimum of the ratio*

$$\frac{\int_{B_R} |x|^{-(N-2)} |\nabla v(|x|)|^2 dx}{\left(\int_{B_R} |x|^{-N} \left(-\log\left(\frac{|x|}{R}\right)\right)^{-\frac{2(N-1)}{N-2}} |v(|x|)|^{\frac{2N}{N-2}} dx\right)^{\frac{N-2}{N}}}, \quad (1.19)$$

is

$$C_{HS} := S(N) (N-2)^{-2(N-1)/N} \quad (1.20)$$

and it is achieved by

$$v_{\mu,\nu}(|x|) = \psi_{\mu,\nu} \left( \left( -\log \left( \frac{|x|}{R} \right) \right)^{-\frac{1}{N-2}} \right) = \left( \mu^2 + \nu^2 \left( -\log \left( \frac{|x|}{R} \right) \right)^{-\frac{2}{N-2}} \right)^{-\frac{N-2}{2}}, \quad (1.21)$$

where the infimum is taken over the radially symmetric functions of  $W_0^{1,2}(|x|^{-(N-2)}, B_R)$ .

Observe that we may continuously define  $u(0) = \mu^{-(N-2)}$  and  $u(R) = 0$ . We also have (see the proof of Theorem 1.1), that  $v_{\mu,\nu} \in W^{1,2}(|x|^{-(N-2)}, B_R)$  but  $|x|^{-(N-2)} v_{\mu,\nu} \notin H_0^1(B_R)$ .

Concerning (1.6), under the transformation (1.17), we relate it with the Sobolev inequality (1.2). Then, we prove that the best constant in (1.6) is  $C_{HS}$ , as defined in (1.20) and the minimizers of (1.6) are

$$u_{m,n}(|x|) = |x|^{-\frac{N-2}{2}} \psi_{m,n} \left( \left( -\log \left( \frac{|x|}{R} \right) \right)^{-\frac{1}{N-2}} \right), \quad x \in B_R \setminus \{0\}, \quad \phi_n|_{\partial B_R} = 0. \quad (1.22)$$

where  $\psi_{m,n}$  are given by (1.21);

**THEOREM 1.2** *The inequalities (1.6) and*

$$\int_{\mathbb{R}^N} (\nabla w(t))^2 dt \geq (N-2)^{\frac{2(N-1)}{N}} C_{HS} \left( \int_{\mathbb{R}^N} |w(t)|^{\frac{2N}{N-2}} dt \right)^{\frac{N-2}{N}}. \quad (1.23)$$

are equivalent under the transformation (1.17). Then, the best constant in (1.6) is (1.20) and the minimizers are given by (1.22).

In this direction, making some straightforward calculations, we have that

**THEOREM 1.3** *For each  $n$ ,  $\phi_n$  solves the corresponding to (1.6) Euler-Lagrange equation:*

$$\begin{aligned} -\Delta u - \left( \frac{N-2}{2} \right)^2 \frac{u}{|x|^2} &= \left( -\log \left( \frac{|x|}{R} \right) \right)^{-\frac{2(N-1)}{N-2}} u^{\frac{N+2}{N-2}} \\ u|_{\partial\Omega} &= 0. \end{aligned} \quad (1.24)$$

For the nonradial case, i.e.  $\Omega$  is an arbitrary bounded domain in  $\mathbb{R}^N$ , containing the origin,  $D_0 = \sup_{x \in \Omega} D$  and  $D > D_0$ , we refer to the recent work [3].

The paper is organized as follows: In Section 2 we consider the spaces  $W_0^{1,2}(|x|^{-(N-2)}, \Omega)$  and  $H(\Omega)$ , we prove Lemma 1.2 and as a consequence we obtain some Caffarelli-Kohn-Nirenberg inequalities. In Section 3 we consider inequality (1.1) and in Section 3 we give the proof of the remaining theorems.

For Hardy inequalities and their possible improvements we refer to [11, 12, 13, 14, 21, 22, 23, 25, 32, 37] and for various type of Hardy-Sobolev inequalities we refer to the works [2, 4, 5, 8, 7, 16, 19, 24, 26, 27, 31, 34, 35].

**Notation** In the sequel we often use the notation  $r = |x|$ .

## 2 The spaces $W_0^{1,2}(|x|^{-(N-2)}, \Omega)$ and $H(\Omega)$

In this section we give some further properties for the spaces  $W_0^{1,2}(|x|^{-(N-2)}, \Omega)$  and  $H(\Omega)$  and give the connection between them, i.e., we give the proof of Lemma 1.2.

Concerning  $W_0^{1,2}(|x|^{-(N-2)}, \Omega)$  from [25, Lemma 2.1] we have that

- LEMMA 2.1** (i) If  $u \in H_0^1(\Omega)$ , then  $|x|^{\frac{N-2}{2}}u \in W_0^{1,2}(|x|^{-(N-2)}, \Omega)$ .  
(ii) If  $v \in W_0^{1,2}(|x|^{-(N-2)}, \Omega)$ , then  $|x|^{-a}v \in H_0^1(\Omega)$ , for all  $a < \frac{N-2}{2}$ .  
(iii) The norm

$$\left( \int_{\Omega} |x|^{-(N-2)} |\nabla w|^2 dx + \int_{\Omega} |x|^{-(N-2)} w^2 dx \right)^{1/2}$$

is an equivalent norm for the space  $W_0^{1,2}(|x|^{-(N-2)}, \Omega)$ .

- (iv) The space  $W_0^{1,2}(|x|^{-(N-2)}, \Omega)$  is a Hilbert space with inner product

$$\langle \phi, \psi \rangle_{W_0^{1,2}(|x|^{-(N-2)}, \Omega)} = \int_{\Omega} |x|^{-(N-2)} \nabla \phi \cdot \nabla \psi dx.$$

Concerning  $H(\Omega)$  from [37] we have that

- LEMMA 2.2** (i) The space  $H(\Omega)$  is a Hilbert space with inner product

$$\langle \phi, \psi \rangle_{H(\Omega)} = \int_{\Omega} \nabla \phi \cdot \nabla \psi dx - \left( \frac{N-2}{2} \right)^2 \int_{\Omega} \frac{\phi \psi}{|x|^2} dx - L(\phi) L(\psi).$$

- (ii) If  $u \in H_0^1(\Omega)$ , then  $u \in H(\Omega)$  and if  $u \in H(\Omega)$  then  $u \in \cap_{q < 2} W^{1,q}(\Omega)$ , i.e.,

$$H_0^1(\Omega) \subset H(\Omega) \subset \cap_{q < 2} W^{1,q}(\Omega)$$

- (iii) The continuous imbedding  $H(\Omega) \hookrightarrow H_0^s$ ,  $0 \leq s < 1$  imply that the space  $H(\Omega)$  is compactly embedded in  $L^q(\Omega)$ , for any  $1 \leq q < \frac{2N}{N-2}$ .

Moreover, from [37, Theorem 4.2] we have the following.

**THEOREM 2.1** Let  $B_R$  the sphere in  $\mathbb{R}^N$ ,  $N \geq 3$ , centered at the origin with radius  $R$ . Let  $z_{m,n}$  be the  $n$ -th zero of the Bessel function  $J_m$  and  $\phi_k(\sigma)$  be the orthonormal eigenfunctions of the Laplace-Beltrami operator with corresponding eigenvalues  $c_k = k(N+k-2)$ ,  $k \geq 0$ . Then, the two-parameter family

$$e_{k,n}(r, \sigma) = r^{-\frac{N-2}{2}} J_m \left( \frac{z_{m,n}}{R} r \right) \phi_k(\sigma), \quad (2.1)$$

with  $m^2 = k(k+N-2)$ , consist an orthogonal basis of  $L^2(B_R)$ .

**REMARK 2.1** Note that all the  $J_m$  vanish at  $r = 0$ , except  $J_0$  for which (under normalization)  $J_0(0) = 1$ . Then, the maximal singularity corresponds to the sub-family of eigenfunctions with  $j = 0$

$$e_{0,n} = O(r^{-\frac{N-2}{2}})$$

These functions represent the complete sub-basis for the subspace  $X_1$  of radial functions in  $L^2(B_R)$ . They do not belong to  $H_0^1(\Omega)$  but belong to  $H(\Omega)$ .

**Proof of Lemma 1.2** (i) Let  $v \in C_0^\infty(\Omega)$ , setting  $u(x) = |x|^{-(N-2)/2} v(x)$  we have

$$\int_{\Omega} |\nabla u|^2 dx - \left( \frac{N-2}{2} \right)^2 \int_{B_R} \frac{u^2}{|x|^2} dx = \int_{\Omega} |x|^{-(N-2)} |\nabla v|^2 dx + \frac{1}{2} \int_{\Omega} \nabla |x|^{-(N-2)} \cdot \nabla v^2 dx. \quad (2.2)$$

We first treat the radial case; we assume that  $\Omega = B_R$  and let  $v(r) \in C^\infty(0, R)$ ,  $v(R) = 0$  and  $v(r) \in W_{0,r}^{1,2}(|x|^{-(N-2)}, \Omega)$ . From this point of view the second integral in the right hand side of (2.2) is equal to

$$\frac{1}{2} \int_{\Omega} \nabla |x|^{-(N-2)} \cdot \nabla v^2 dx = -N \omega_N \frac{N-2}{2} \int_0^R (v^2)' dr = \frac{N(N-2)}{2} \omega_N v^2(0).$$

Then, (2.2) implies that  $u \in H(B_R)$ . For the nonradial case, in order to estimate the second integral in the right hand side of (2.2), we use the decomposition into spherical harmonics; Let  $v \in C_0^\infty(\Omega)$ . If we extend  $u$  as zero outside  $\Omega$ , we may consider that  $v \in C_0^\infty(\mathbb{R}^N)$ . Decomposing  $v$  into spherical harmonics we get

$$v = \sum_{k=0}^{\infty} v_k := \sum_{k=0}^{\infty} f_k(r) \phi_k(\sigma),$$

where  $\phi_k(\sigma)$  are the orthonormal eigenfunctions of the Laplace-Beltrami operator with corresponding eigenvalues  $c_k = k(N+k-2)$ ,  $k \geq 0$ . The functions  $f_k$  belong in  $C_0^\infty(\mathbb{R}^N)$ , satisfying

$$f_k(r) = O(r^k), \quad \text{and} \quad f'_k(r) = O(r^{k-1}), \quad \text{as } r \downarrow 0. \quad (2.3)$$

In particular,  $\phi_0(\sigma) = 1$  and  $v_0(r) = \frac{1}{|\partial B_r|} \int_{\partial B_r} u ds$ , for any  $r > 0$ . Then, for any  $k \in \mathbb{N}$ , from (2.3) we have that

$$\begin{aligned} \frac{1}{2} \int_{\Omega} \nabla |x|^{-(N-2)} \cdot \nabla v^2 dx &= \frac{1}{2} \sum_{k=0}^{\infty} \int_{\mathbb{R}^N} \nabla |x|^{-(N-2)} \cdot \nabla f_k^2 dx \\ &= \frac{N(N-2)}{2} \omega_N \sum_{k=0}^{\infty} \lim_{r \rightarrow 0} f_k^2(r) \\ &= \frac{N(N-2)}{2} \omega_N f_0^2(0) \\ &= \frac{N(N-2)}{2} \omega_N v_0(0). \end{aligned} \quad (2.4)$$

Then,  $u \in H(V)$  and (2.2) is equal to (1.14).

(ii) Assume now that  $u \in H(\Omega)$ . Setting  $v(x) = |x|^{N-2/2} u(x)$  we have

$$\begin{aligned} \int_{\Omega} |x|^{-(N-2)} |\nabla v|^2 dx &= \int_{\Omega} |\nabla u|^2 dx + \left( \frac{N-2}{2} \right)^2 \int_{B_R} \frac{u^2}{|x|^2} dx \\ &\quad + \frac{N-2}{2} \int_{\Omega} |x|^{-2} x \cdot \nabla u^2 dx. \end{aligned} \quad (2.5)$$

In order to estimate the last integral above, we use Theorem 2.1. Since  $u \in H(\Omega)$  from Lemma 2.2 we have that  $u \in L^2(\Omega)$ . Moreover, for some  $R > \sup_{x \in \Omega} |x|$ , we may assume that  $u \in L^2(B_R)$ . Then, Theorem 2.1 implies that there exist  $c_n \in \mathbb{R}$ ,  $n = 1, \dots$  such that

$$u = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} c_n \tilde{e}_{k,n}(r) \phi_k(\sigma),$$

where

$$\tilde{e}_{k,n}(r) := r^{-\frac{N-2}{2}} J_m \left( \frac{z_{m,n}}{R} r \right).$$

Then,

$$\begin{aligned} \frac{N-2}{2} \int_{\Omega} |x|^{-2} x \cdot \nabla u^2 dx &= \frac{N-2}{2} \int_{B_R} |x|^{-2} x \cdot \nabla u^2 dx \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} c_n \frac{N-2}{2} \int_{B_R} |x|^{-2} x \cdot \nabla \tilde{e}_{k,n}^2(r) dx. \end{aligned} \quad (2.6)$$

We have further, for every  $k$  and  $n$ , that

$$\begin{aligned} \frac{N-2}{2} \int_{B_R} |x|^{-2} x \cdot \nabla \tilde{e}_{k,n}^2(r) dx &= \frac{N-2}{2} N \omega_N \int_0^R r^{N-2} (\tilde{e}_{k,n}^2(r))' dr \\ &= -\frac{(N-2)^2}{2} N \omega_N \int_0^R r^{N-3} \tilde{e}_{k,n}^2(r) dr \\ &\quad - \frac{N-2}{2} N \omega_N \lim_{r \rightarrow 0} r^{N-2} \tilde{e}_{k,n}^2(r). \end{aligned}$$

or

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{N-2}{2} c_n \int_{B_R} \nabla |x|^{-2} x \cdot \nabla \tilde{e}_{k,n}^2(r) dx &= -\frac{(N-2)^2}{2} \int_{\Omega} \frac{u^2}{|x|^2} dx \\ &\quad - \frac{N(N-2)}{2} \omega_N \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} c_n \lim_{r \rightarrow 0} r^{N-2} \tilde{e}_{k,n}^2(r). \end{aligned} \quad (2.7)$$

However, Remark 2.1 implies that the only nonzero terms in the above limit is given by  $\tilde{e}_{0,n}^2(r)$ . Hence, (2.6) and (2.7) give that

$$\frac{N-2}{2} \int_{\Omega} \nabla |x|^{-2} x \cdot \nabla u^2 dx = -\frac{(N-2)^2}{2} \int_{\Omega} \frac{u^2}{|x|^2} dx - \frac{N(N-2)}{2} \omega_N c_0.$$

Finally, (2.5) becomes

$$\int_{\Omega} |x|^{-(N-2)} |\nabla v|^2 dx = \int_{\Omega} |\nabla u|^2 dx - \frac{(N-2)^2}{2} \int_{\Omega} \frac{u^2}{|x|^2} dx - \frac{N(N-2)}{2} \omega_N c_0.$$

It is clear from the above discussion that  $c_0$  corresponds to  $v_0(0)$ , i.e. we again derive (1.14) and the proof is completed. ■

**COROLLARY 2.1** *Assume now that  $v_n$  is a bounded sequence in  $W_0^{1,2}(|x|^{-(N-2)}, \Omega)$ . Then  $u_n = |x|^{-(N-2)/2} v_n$  is a bounded sequence in  $H(\Omega)$ . The compact imbeddings of Lemma 2.2 imply that, up to some subsequence,  $u_n$  converge in  $L^q(\Omega)$  to some  $u$ . Thus, we obtain the compact imbeddings*

$$W_0^{1,2}(|x|^{-(N-2)}, \Omega) \hookrightarrow L^q(|x|^{-q(N-2)/2}, \Omega), \quad \text{for any } 1 \leq q < \frac{2N}{N-2} \quad (2.8)$$

and since  $1 \leq q$ , we further obtain the compact imbeddings

$$W_0^{1,2}(|x|^{-(N-2)}, \Omega) \hookrightarrow L^q(|x|^{-(N-2)/2}, \Omega), \quad \text{for any } 1 \leq q < \frac{2N}{N-2}, \quad (2.9)$$



where the weighted space  $L^q(w(x), \Omega)$  is defined as the closure of  $C_0^\infty(\Omega)$  functions under the norm

$$\|\phi\|_{L^q(w(x), \Omega)} = \left( \int_{\Omega} w(x) |\phi|^q dx \right)^{\frac{1}{q}}.$$

**REMARK 2.2** In (2.8) is clear that  $q$  cannot reach  $\frac{2N}{N-2}$ . For this value of  $q$  the best that we can have is the inequality corresponding to (1.19). In this sense the results obtained in the previous Corollary complete the results obtained in [15] (see also [17, 18, 38]) concerning the Caffarelli - Kohn - Nirenberg Inequalities, in the limiting case where  $a = \frac{N-2}{2}$ .

### 3 Inequalities (1.1) and (1.6)

In [29, Theorem 4] the following general inequality was proved

$$\left[ \int_{-\infty}^{+\infty} \left| \int_r^{\infty} f(t) dt \right|^q d\mu(r) \right]^{1/q} \leq C \left[ \int_{-\infty}^{+\infty} |f(r)|^p d\nu(r) \right]^{1/p}, \quad (3.1)$$

where  $1 \leq p \leq q \leq \infty$ , which holds for any  $f \in C_0^\infty(\mathbb{R})$ , if and only if the following quantity

$$B = \sup_{l \in (-\infty, +\infty)} [\mu((-\infty, l))]^{1/q} \left[ \int_l^{\infty} \left( \frac{d\nu^*}{dr} \right)^{-1/(p-1)} dr \right]^{(p-1)/p},$$

where  $\nu^*$  is the absolutely continuous part of  $\nu$ , is finite. Moreover, if  $C$  is the best constant in (3.1), then

$$B \leq C \leq B \left( \frac{q}{q-1} \right)^{(p-1)/p} q^{1/q}. \quad (3.2)$$

Inequality (1.1) is obtained (see [25, Lemma 2.2]) by setting  $p = 2$ ,  $q = 2N/(N-2)$ ,  $f(r) = v'(r)$ ,  $\mu(r) = r^{-1} (-\log r)^{-2(N-1)/(N-2)} \chi_{(0,1)} dr$  and  $d\nu = r \chi_{(0,1)} dr$

As prof. V Maz'ya pointed to us inequality (1.1) may also be obtained from Bliss' inequality:

**PROPOSITION 3.1** For all  $v : (0, \infty) \rightarrow \mathbb{R}$  absolutely continuous with  $v' \in L^k(0, \infty)$  and  $v(0) = 0$  one has

$$\int_0^\infty \frac{|v|^l}{|x|^{l-h}} dx \leq K \left( \int_0^\infty |v'|^k dx \right)^{l/k}, \quad (3.3)$$

where  $l > k > 1$ ,  $h = l/k - 1$  and

$$K = \frac{1}{l-h-1} \left[ \frac{h \Gamma(l/h)}{\Gamma(1/h) \Gamma((l-l)/h)} \right]^h.$$

Moreover, equality holds in (3.3) if and only if

$$v(x) = (a + bx^{-h})^{-1/h}, \quad (3.4)$$

for arbitrary positive constants  $a$  and  $b$ .

In the case now where  $k = 2$  and  $l = 2N/(N - 2)$  we have that  $h = 2/(N - 2)$  and  $l - h = 2(N - 1)/(N - 2)$ . Hence (3.3) is equal to

$$\int_0^\infty |v(t)|^{\frac{2N}{N-2}} t^{-\frac{2(N-1)}{N-2}} dt \leq K \left( \int_0^\infty |v'(t)|^2 dt \right)^{N/(N-2)}. \quad (3.5)$$

As an alternative proof of Lemma 1.1 we may prove the following.

**LEMMA 3.1** (a) *Inequality (3.5) under the change of variables*

$$t = -\log\left(\frac{r}{R}\right)$$

is equivalent to (1.1)

(b) *Inequality (3.5) under the change of variables*

$$t = r^{-(N-2)}$$

is equivalent to (1.2).

We now give the proof of Lemma 1.1.

**Proof of Lemma 1.1** Let  $v \in C_0^\infty(0, R)$ . Using the transformation (1.3) we have that

$$v'(r) = \frac{1}{N-2} w'(t) t^{N-1} r^{-1}$$

and

$$dr = (N-2) t^{-(N-1)} r.$$

Then,

$$\int_0^R r |v'|^2 dr = \int_0^\infty r \frac{1}{N-2} |w'(t)|^2 t^{N-1} r^{-1} dt = \frac{1}{N-2} \int_0^\infty t^{N-1} |w'(t)|^2 dt$$

and

$$\begin{aligned} \int_0^R r^{-1} \left(-\log\left(\frac{r}{R}\right)\right)^{-\frac{2(N-1)}{N-2}} |v|^{\frac{2N}{N-2}} dr &= \int_0^\infty r^{-1} t^{2(N-1)} |w(t)|^{\frac{2N}{N-2}} (N-2) t^{-(N-1)} r dt \\ &= (N-2) \int_0^\infty t^{N-1} |w(t)|^{\frac{2N}{N-2}} dt. \end{aligned}$$

So, inequality (1.1) becomes

$$\frac{1}{N-2} \int_0^\infty t^{N-1} |w'(t)|^2 dt \geq c(N-2)^{\frac{N-2}{N}} \left( \int_0^\infty t^{N-1} |w(t)|^{\frac{2N}{N-2}} dt \right)^{\frac{N-2}{N}}$$

or

$$\int_{\mathbb{R}^N} |\nabla w|^2 dx \geq c(N-2)^{\frac{2(N-1)}{N}} (N \omega_N)^{\frac{2}{N}} \left( \int_{\mathbb{R}^N} |w|^{\frac{2N}{N-2}} dx \right)^{\frac{N-2}{N}}.$$

It is clear that if  $v \in C_0^\infty(0, R)$  we have that  $w \in D^{1,2}(\mathbb{R}^N)$ . Then, the best constant and the minimizers are given by (1.4) and (1.5), respectively and the proof is completed. ■

**Proof of Lemma 1.2** Follows directly from Lemma 1.1 and Lemma 1.2. ■

**Acknowledgements.** The author thanks prof. V. Maz'ya for noticing the connection between inequality (1.1) and the Bliss inequality, prof. Luis Escauriaza for bringing to his attention the work [30] and prof. A. Tertikas for informing, after a personal communication, the results that were obtained in [3].

## References

- [1] R. A. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.
- [2] Adimurthi, and M. J. Esteban, *An Improved Hardy-Sobolev Inequality in  $W^{1,p}$  and its Application to Schrödinger Operators*, NODEA, **12**, (2005), no. 2, 243-263.
- [3] Adimurthi, S. Filippas and A. Tertikas *On the Best Constant of Hardy-Sobolev Inequalities*, Nonlinear An. TMA, to appear.
- [4] Adimurthi and A. Sekar, *Role of the fundamental solution in Hardy-Sobolev-type inequalities*, Proc. Roy. Soc. Edinburgh Sect. A, **136**, (2006), no. 6, 1111-1130.
- [5] A. Alvino, V. Ferone and G. Trombetti, *On the Best Constant in a Hardy-Sobolev Inequality*, Applicable Analysis, **85**, (2006), no. 1-3, 171-180.
- [6] T. Aubin *Problèmes isopérimétriques et espaces de Sobolev*, J. Differential Geometry, **11**, (1976), no. 4, 573-598.
- [7] A. Balinsky, W. D. Evans, D. Hundertmark, R. T. Lewis, *On inequalities of Hardy-Sobolev type*, Banach J. Math. Anal., **2** (2008), no. 2, 94-106.
- [8] M. Badiale and G. Tarantello, *A Sobolev-Hardy Inequality with Applications to a Nonlinear Elliptic Equation arising in Astrophysics*, Arch. Rational Mech. Anal., **163** (2002), 259-293.
- [9] R. D. Benguria, R. L. Frank and M. Loss, *The Sharp Constant in the Hardy-Sobolev-Maz'ya Inequality in the three Dimensional Upper Half Space*, Math. Res. Lett., **15**, (2008), 613-622.
- [10] G. Bliss, *An integral inequality*, J. London Math. Soc., (5), (1930), 40-46.
- [11] H. Brezis and M. Marcus, *Hardy's inequality revisited*, Ann. Sc. Norm. Pisa, **25**, (1997), 217-237.
- [12] H. Brezis, M. Marcus and I. Shafrir, *Extremal functions for Hardy's inequality with weight*, J. Funct. Anal., **171**, (2000), 177-191.
- [13] H. Brezis and J. L. Vázquez, *Blowup solutions of some nonlinear elliptic problems*, Revista Mat. Univ. Complutense Madrid, **10**, (1997), 443-469.
- [14] X. Cabré and Y. Martel, *Existence versus explosion instantanée pour des équations de la chaleur linéaires avec potentiel singulier*, C. R. Acad. Sci. Paris, **329**, Série I, (1999), 973-978.
- [15] L. Caffarelli, R. Kohn and L. Nirenberg, *First Order Interpolation Inequalities with Weights*, Compositio Math., **53**, (1984), no 3., 259-275.
- [16] D. Castorina, I. Fabbri, G. Mancini and K. Sandeep, *Hardy-Sobolev extremals, hyperbolic symmetry and scalar curvature equations*, J. Differential Equations, **246**, (2009), 1187-1206.
- [17] F. Catrina and Z.-Q. Wang, *On the Caffarelli-Kohn-Nirenberg Inequalities: Sharp Constants, Existence (and Nonexistence) and Symmetry of Extremal Functions*, Comm. Pure Appl. Math., **LIV**, (2001), 229-258.
- [18] K. S. Chou and C. W. Chu, *On the Best Constant For a Weighted Sobolev-Hardy Inequality*, J. London Math. Soc. (2), **48**, (1993), no. 1, 137-151.
- [19] N. Chaudhuri, *Bounds for the Best Constant in an Improved Hardy-Sobolev Inequality*, Z. Anal. Anwendungen, **22**, (2003), no. 4, 757-765.
- [20] A. Cianchi, N. Fusco, F. Maggi and A. Pratelli *The Sharp Sobolev Inequality in Quantitative Form*, preprint.
- [21] E. B. Davies, *A review of Hardy inequalities*, Oper. Theory Adv. Appl., **110** (1999), 55-67.
- [22] J. Dávila and L. Dupaigne, *Hardy-type Inequalities*, J. Eur. Math. Soc., **6**, (2004), no 3, 335-365.

- [23] J. Dolbeault, M. J. Esteban, M. Loss and L. Vega, *An Analytical Proof of Hardy-like Inequalities Related to the Dirac Operator*, J. Funct. Anal., **216**, (2004), no 1, 121.
- [24] S. Filippas, V. Maz'ya and A. Tertikas, *Critical Hardy-Sobolev Inequalities*, J. Math. Pure Appl. **87** (2006), 37-56.
- [25] S. Filippas and A. Tertikas, *Optimizing improved Hardy inequalities*, J. Funct. Analysis **192** (2002), 186-233.
- [26] N. Ghoussoub, X. S. Kang, *Hardy-Sobolev critical elliptic equations with boundary singularities*, Ann. Inst. H. Poincare Anal. Non Lineaire, 21, (2004), no. 6, 767-793
- [27] N. Ghoussoub, F. Robert, *The effect of curvature on the best constant in the Hardy-Sobolev inequalities*, Geom. Funct. Anal., 16, (2006), no. 6, 1201-1245.
- [28] E. Lieb and W. Thirring *Inequalities for the moments of the eigenvalues of the Schrödinger equation and their relation to Sobolev inequalities*. Studies in Math. Phys. Essays in honor of V. Bargmann, Princeton University Press, 1976.
- [29] V. G. Maz'ja, Sobolev spaces, Springer-Verlag, 1985.
- [30] Roberta Musina, *A note on the paper [25]*, J. Funct. Analysis, to appear.
- [31] A. I. Nazarov, *Hardy-Sobolev Inequalities in a cone*, J. Math. Sciences, **132**, (2006), (4), 419-427.
- [32] B. Opic, and A. Kufner, Hardy type inequalities, Pitman Research Notes in Math., **219**, Longman 1990.
- [33] G. Talenti *Best constant in Sobolev inequality*, Ann. Mat. Pura Appl., **110** (1976), 353-372.
- [34] A. Tertikas and K. Tintarev, *On Existence of Minimizers for the Hardy-Sobolev-Maz'ya Inequality*, Ann. Mat. Pura Appl. (4), **186**, (4), (2007), 645-662.
- [35] A. Tertikas, N. B. Zographopoulos, *Best constants in the Hardy-Rellich inequalities and related improvements*, Adv. Math. **209** (2007), no. 2, 407-459.
- [36] J. L. Vazquez and N. B. Zographopoulos, *A note on the paper "The Hardy Inequality and the Asymptotic Behaviour of the Heat Equation with an Inverse-Square Potential" by Juan Luis Vazquez and Enrike Zuazua*, in preparation.
- [37] J. L. Vazquez and E. Zuazua, *The Hardy inequality and the asymptotic behaviour of the heat equation with an inverse-square Potential*, J. Functional Analysis **173** (2000), 103-153.
- [38] Z.-Q. Wang and M. Willem, *Caffarelli-Kohn-Nirenberg Inequalities with Remainder Terms*, J. Funct. Analysis, **203**, (2003), 550-568.